

Thermodynamics of noncommutative BTZ black hole

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ABSTRACT: Thermodynamics of the BTZ black hole in noncommutative geometry is studied. We work out the Hawking temperature and entropy which reduce to their commutative limits when the noncommutativity parameter tends to zero. We also discuss the range of validity of the Hawking area law in the noncommutative case and provide graphical analysis. We see that the law is not valid unless the outer horizon is very large.

The interest in noncommutativity was developed when it was shown that the field theory becomes noncommutative when matter decouples from gravity, and the spacetime induces a noncommutative coordinate algebra. In loop quantum gravity as well the spatial operators do not commute and one obtains a noncommutative manifold. The noncommutativity is introduced by means of [1]

$$[x^i, x^j] = i\theta^{ij}, \quad (1)$$

where θ^{ij} is antisymmetric tensor, $D \times D$ matrix, where D is the dimension of the spacetime. It has dimensions of $(\text{length})^2$ and it discretizes the spacetime. The noncommutative relation given by Eq. (1) induces quantum mechanical fluctuations in the metric, $g_{\mu\nu}$. The spacetime is quantized and the coordinates become noncommutative. In one approach to noncommutative geometry the point-wise multiplication of fields in the Lagrangian is replaced by the Moyal product given by [2]

$$(f \star g)(x) = \exp\left[\frac{i}{2}\theta^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\right]f(x)g(x), \quad (2)$$

where, f and g are functions of the coordinates. This is a powerful method for studying field theories on noncommutative spaces. As the parameter θ varies from zero to a positive value we go from the commutative to the noncommutative regime. Recently there have been some investigations on noncommutative black holes [1, 2, 3, 4]. Apart from some canonical solutions on noncommutative spaces, such as that of the noncommutative Schwarzschild black hole [1], most of the solutions are not obtained directly from Einstein's field equations.

P. Nicolini *et al* [1] introduced noncommutativity in the Schwarzschild black hole by taking the mass density to be a Gaussian distribution

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}}e^{-\frac{r^2}{4\theta}}, \quad (3)$$

with minimal width $\sqrt{\theta}$, instead of the Dirac delta function, and where the noncommutative parameter, θ is a small positive number of the order of $(\text{Planck length})^2$. Using Eq. (3) one can write the mass of the noncommutative Schwarzschild black hole of radius r in the following way [1]

$$m_\theta(r) = \int_0^r 4\pi\dot{r}^2 \rho_\theta(\dot{r}) d\dot{r} = \frac{2M}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \quad (4)$$

where M is mass of the commutative Schwarzschild black hole and $\gamma(\frac{3}{2}, \frac{r^2}{4\theta})$ is the lower incomplete gamma function. In Einstein's equations using the energy-momentum tensor

$$(T_\theta)_b^a = \text{diag}[-\rho_\theta, p_r, p', p'], \quad (5)$$

where $p_r = -\rho_\theta$ and $p' = p_r - \frac{r}{2}\partial_r \rho_\theta$ are pressure terms, the noncommutative Schwarzschild solution is [1]

$$ds^2 = -f_\theta(r)dt^2 + \frac{dr^2}{f_\theta(r)} + r^2 d\Omega^2, \quad (6)$$

where $f_\theta(r) = 1 - \frac{4M}{r\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{r^2}{4\theta})$ and $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2$. It is same as if we replace the mass term in the metric with the noncommutative mass term given in Eq. (4).

Banados *et al* [5] discovered a black hole solution of Einstein's equations with a negative cosmological constant, in $(2+1)$ dimensions in the commutative spacetime. The noncommutative Banados-Teitelboim-Zanelli (BTZ) metric [2, 4] was found by using the Chern-Simons theory which is a 3-dimensional topological quantum field theory. In this theory the action for the field A is proportional to the integral of the Chern-Simons 3-form i.e.

$$S(A) = \beta \int \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (7)$$

where $\beta = l/16\pi G$ and G is Newton's constant. Classically the system is characterized by its equations of motion which are the extrema of the action with respect to variations of the field, A , and are given by

$$F = dA + A \wedge A. \quad (8)$$

For the $(2+1)$ -dimensional noncommutative Chern-Simons theory with negative cosmological constant $-1/l^2$ the action is given by [2, 4]

$$\hat{S}(\hat{A}^+, \hat{A}^-) = \hat{S}_+(\hat{A}^+) - \hat{S}_-(\hat{A}^-), \quad (9)$$

$$\hat{S}_\pm(\hat{A}^\pm) = \beta \int \text{Tr}(\hat{A}^\pm \overset{\star}{\wedge} d\hat{A}^\pm + \frac{2}{3} \hat{A}^\pm \overset{\star}{\wedge} \hat{A}^\pm \overset{\star}{\wedge} \hat{A}^\pm). \quad (10)$$

The deformed wedge product is defined as

$$A \overset{\star}{\wedge} B = A_a \star B_b dx^a \wedge dx^b, \quad (11)$$

where \star is the Moyal product defined in Eq. (2). The noncommutative $U(1, 1) \times U(1, 1)$ gauge fields, \tilde{A} , consist of noncommutative $SU(1, 1) \times SU(1, 1)$ gauge fields, \hat{A} and two $U(1)$ fluxes, \hat{B} [2, 4]

$$\tilde{A}^\pm = \tilde{A}^{A\pm} \tau_A = \hat{A}^{a\pm} \tau_a + \hat{B}^\pm \tau_3. \quad (12)$$

The noncommutative $SU(1, 1) \times SU(1, 1)$ gauge fields \hat{A} are expressed in terms of the triad \hat{e} and the spin connection \hat{w} as

$$\hat{A}^{a\pm} = \hat{w}^a \pm \frac{\hat{e}^a}{l}. \quad (13)$$

Using the original Seiberg-Witten map [6] and the commutation relation $[R, \phi] = 2i\theta$, where $R = r^2$, the solution will be

$$\tilde{A}_\mu^\pm(A) = (A_\mu^{a\pm} - \frac{\theta}{2} B^\pm \partial_R A_\mu^{a\pm}) \tau_a + B^\pm \tau_3. \quad (14)$$

The commutative gauge fields are given by [4]

$$A^{0\pm} = \pm \frac{m(r_+ \pm r_-)}{l^2} (dt \mp l d\phi), \quad (15)$$

$$A^{1\pm} = \pm \frac{dm}{n}, \quad (16)$$

$$A^{2\pm} = -\frac{n(r_+ \pm r_-)}{l^2} (dt \mp l d\phi). \quad (17)$$

Using these in Eq. (14), the noncommutative gauge fields can be written as [4]

$$\hat{A}^{0\pm} = -\frac{(m - \frac{\theta\beta}{2} m')(r_+ \pm r_-)}{l} (d\phi \mp \frac{dt}{l}), \quad (18)$$

$$\hat{A}^{1\pm} = \pm [\frac{m'}{n} - \frac{\theta\beta}{2} (\frac{m'}{n})'] dR, \quad (19)$$

$$\hat{A}^{2\pm} = \pm \frac{(n - \frac{\theta\beta}{2} n')(r_+ \pm r_-)}{l} (d\phi \mp \frac{dt}{l}). \quad (20)$$

Using Eq. (13) the triad and the spin connection become

$$\hat{e}^0 = (m - \frac{\theta\beta}{2} m') (\frac{r_+}{l} dt - r_- d\phi) + \text{Order}(\theta^2), \quad (21)$$

$$\hat{e}^1 = \pm [\frac{m'}{n} - \frac{\theta\beta}{2} (\frac{m'}{n})'] dR + \text{Order}(\theta^2), \quad (22)$$

$$\hat{e}^2 = (n - \frac{\theta\beta}{2}n')(r_+d\phi - \frac{r_-}{l}dt) + \text{Order}(\theta^2), \quad (23)$$

$$\hat{w}^0 = -\frac{(m - \frac{\theta\beta}{2}m')}{l}(r_+d\phi - \frac{r_-}{l}dt) + \text{Order}(\theta^2), \quad (24)$$

$$\hat{w}^1 = \text{Order}(\theta^2), \quad (25)$$

$$\hat{w}^2 = -\frac{(n - \frac{\theta\beta}{2}n')}{l}(\frac{r_+}{l}dt - r_-d\phi) + \text{Order}(\theta^2), \quad (26)$$

where prime denotes differentiation with respect to $R = r^2$. The metric in the noncommutative case is given as

$$d\hat{s}^2 = \eta_{ab}\hat{e}_\mu^a \star \hat{e}_\nu^b dx^\mu dx^\nu. \quad (27)$$

Since the metric coefficients are only R -dependent so $\hat{e}_\mu^a \star \hat{e}_\nu^b = \hat{e}_\mu^a \hat{e}_\nu^b$, and we have

$$ds^2 = -\hat{f}^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 \hat{N}^\phi dt d\phi + (r^2 - \frac{\theta\beta}{2}) d\phi^2 + \text{Order}(\theta^2). \quad (28)$$

Here

$$\begin{aligned} \hat{f}^2 &= \frac{r^2 - r_+^2 - r_-^2}{l^2} - \frac{\theta\beta}{2}, \\ \hat{N}^2 &= \frac{1}{r^2 l^2} \left\{ (r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta\beta}{2} (2r^2 - r_+^2 - r_-^2) \right\}, \\ \hat{N}^\phi &= \frac{-r_+ r_-}{l r^2}, \end{aligned} \quad (29)$$

where r_+ and r_- are the outer and inner horizons of the commutative BTZ black hole [5]

$$r_\pm^2 = \frac{l^2 M}{2} \left\{ 1 \pm \left(1 - \left(\frac{J}{Ml} \right)^2 \right)^{\frac{1}{2}} \right\}. \quad (30)$$

The apparent horizons, for the noncommutative black hole, denoted by \hat{r}_\pm can be determined by $\hat{N}^2 = 0$

$$\hat{r}_\pm^2 = r_\pm^2 + \frac{\beta\theta}{2} + \text{Order}(\theta^2). \quad (31)$$

This shows that the event horizons in the noncommutative case, are shifted through constant $\beta\theta/2$. Both the inner and outer horizons are equally shifted. In the limit $\theta \rightarrow 0$, these reduce to the event horizons of the commutative case.

Let us rewrite the metric of noncommutative BTZ black hole as

$$ds^2 = -f dt^2 + g^{-1} dr^2 + (r^2 - \frac{\beta\theta}{2}) d\chi^2, \quad (32)$$

where

$$\begin{aligned} f &= \left(\frac{r^2 - Ml^2}{l^2} + \frac{J^2}{4(r^2 - \frac{\beta\theta}{2})} - \frac{\beta\theta}{2} \right), \\ g &= \frac{r^2 - Ml^2}{l^2} + \frac{J^2}{4r^2} - \frac{\beta\theta}{2} \left(\frac{2}{l^2} - \frac{M}{r^2} \right), \\ d\chi &= d\phi - \frac{J}{2(r^2 - \frac{\beta\theta}{2})} dt. \end{aligned}$$

There is a coordinate singularity at $g(\hat{r}_+) = 0$. This singularity is removed by the use of the Painleve-type coordinate transformation [7]

$$dt \rightarrow dt - \sqrt{\left(\frac{1-g}{fg} \right)} dr.$$

Using this transformation in metric (32) we get

$$ds^2 = -f dt^2 + dr^2 + 2f \sqrt{\left(\frac{1-g}{fg} \right)} dr dt + (r^2 - \frac{\beta\theta}{2}) d\chi^2. \quad (33)$$

Near the outer horizon we expand the functions f and g using Taylor's series

$$f(\hat{r}_+) = f'(\hat{r}_+)(r - \hat{r}_+) + O((r - \hat{r}_+)^2), \quad (34)$$

$$g(\hat{r}_+) = g'(\hat{r}_+)(r - \hat{r}_+) + O((r - \hat{r}_+)^2). \quad (35)$$

The surface gravity at the outer horizon is given by

$$\begin{aligned} \kappa &= \Gamma_{00}^0|_{r=r_+}, \\ &= \frac{1}{2} \left[\sqrt{\frac{1-g}{fg}} g \frac{df}{dr} \right] \Big|_{r=r_+}. \end{aligned} \quad (36)$$

In our case this becomes

$$\kappa = \frac{1}{2} (\sqrt{f'(\hat{r}_+)g'(\hat{r}_+)}).$$

The Hawking temperature, $T = \hbar\kappa/2\pi$, at the outer horizon takes the form

$$T_h = \frac{1}{4\pi} \hbar \sqrt{f'(\hat{r}_+) g'(\hat{r}_+)}. \quad (37)$$

The mass M of the noncommutative black hole is given by

$$M = \frac{r^2}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\beta\theta}{l^2} \right) \Big|_{r=\hat{r}_+}. \quad (38)$$

If we take limit $\theta \rightarrow 0$ we recover the mass of the commutative BTZ black hole. Thus the temperature is obtained as

$$T = \frac{\hbar}{4\pi} \sqrt{\left(\frac{2r}{l^2} - \frac{J^2 r}{2(r^2 - \frac{\beta\theta}{2})^2} \right) \left(\frac{2r}{l^2} - \frac{J^2}{2r^3} - \frac{\beta\theta}{r^3} \left(\frac{r^2}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\beta\theta}{l^2} \right) \right) \right)} \Big|_{r=\hat{r}_+}. \quad (39)$$

Again note that in the limit $\theta \rightarrow 0$ this reduces to the temperature of the commutative BTZ black hole [8]. The first law of black hole thermodynamics is given by

$$dS = \frac{dM}{T} - \frac{\Omega}{T} dJ. \quad (40)$$

At the outer horizon $r = \hat{r}_+$ we have

$$M = M(\hat{r}_+, J), \quad (41)$$

$$dM = \frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+ + \frac{\partial M}{\partial J} dJ. \quad (42)$$

But $\frac{\partial M}{\partial J} = \Omega$ [3] so

$$dM = \frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+ + \Omega dJ. \quad (43)$$

Now using this in Eq. (40) we get

$$d\hat{S} = \frac{1}{T} \frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+, \quad (44)$$

where \hat{S} is the entropy of noncommutative BTZ black hole. Putting Eqs. (38) and (39) in Eq. (44) we get

$$d\hat{S} = \frac{4\pi}{\hbar} \frac{\left[-\frac{2r}{(r^2 - \frac{\beta\theta}{2})^2} \left(\frac{r^4}{l^2} + \frac{J^2}{4} - \frac{\beta\theta r^2}{l^2} \right) + \frac{1}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{4r^3}{l^2} - \frac{2\beta\theta r}{l^2} \right) \right]}{\sqrt{\left(\frac{2r}{l^2} - \frac{J^2 r}{2(r^2 - \frac{\beta\theta}{2})^2} \right) \left(\frac{2r}{l^2} - \frac{J^2}{2r^3} - \frac{\beta\theta}{r^3} \left(\frac{r^2}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\beta\theta}{l^2} \right) \right) \right)}} dr \Big|_{r=\hat{r}_+}. \quad (45)$$

Taking $\hbar = 1$ and integrating yields

$$\hat{S} = 4\pi \int \frac{\left[-\frac{2r}{(r^2 - \frac{\beta\theta}{2})^2} \left(\frac{r^4}{l^2} + \frac{J^2}{4} - \frac{\beta\theta r^2}{l^2} \right) + \frac{1}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{4r^3}{l^2} - \frac{2\beta\theta r}{l^2} \right) \right]}{\sqrt{\left(\frac{2r}{l^2} - \frac{J^2 r}{2(r^2 - \frac{\beta\theta}{2})^2} \right) \left(\frac{2r}{l^2} - \frac{J^2}{2r^3} - \frac{\beta\theta}{r^3} \left(\frac{r^2}{(r^2 - \frac{\beta\theta}{2})} \left(\frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\beta\theta}{l^2} \right) \right) \right)}} dr \Big|_{r=\hat{r}_+}. \quad (46)$$

If we set $\theta = 0$ in Eq. (46), we recover the standard formula for the entropy of the commutative BTZ black hole [8]

$$\begin{aligned} S &= 4\pi \int dr|_{r=\hat{r}_+}, \\ S &= 4\pi r_+. \end{aligned} \quad (47)$$

In order to integrate Eq. (46) to find entropy in the noncommutative case we rewrite the numerator and the denominator of the integrand as

$$\text{Numerator} = \frac{2r}{l^2} \left(1 - \frac{J^2 l^2}{4r^4} \right) \left(1 - \frac{\beta\theta J^2 l^2}{r^2(4r^4 - J^2 l^2)} \right) + \text{Order}(\theta^2), \quad (48)$$

$$\begin{aligned} \text{Denominator} &= \frac{2r}{l^2} \left(1 - \frac{J^2 l^2}{4r^4} \right) \sqrt{\left(1 - \frac{\beta\theta J^2 l^2}{r^2(4r^4 - J^2 l^2)} \right) \left(1 - \beta\theta \left(\frac{1}{2} + \frac{J^2 l^2}{8r^4} \right) \frac{4r^2}{4r^4 - J^2 l^2} \right)} \\ &\quad + \text{Order}(\theta^2), \end{aligned} \quad (49)$$

so that ignoring the higher order terms in θ Eq. (46) becomes

$$\hat{S} = 4\pi \int \left(1 - \frac{\beta\theta J^2 l^2}{r^2(4r^4 - J^2 l^2)} \right)^{\frac{1}{2}} \left(1 - \beta\theta \left(\frac{1}{2} + \frac{J^2 l^2}{8r^4} \right) \frac{4r^2}{4r^4 - J^2 l^2} \right)^{-\frac{1}{2}} dr \Big|_{r=\hat{r}_+}. \quad (50)$$

On integrating, this can be written as

$$\hat{S} = 4\pi \left(\hat{r}_+ - \frac{\beta\theta}{\hat{r}_+} \right), \quad (51)$$

or, on using Eq. (31), we can write it as

$$\hat{S} = 4\pi \left(r_+ + \frac{\beta\theta}{4} - \frac{\beta\theta}{r_+ + \beta\theta/4} \right), \quad (52)$$

which is the entropy of the noncommutative BTZ black hole. Note that the term

$$4\pi \left(\frac{\beta\theta}{4} - \frac{\beta\theta}{r_+ + \beta\theta/4} \right)$$

is the correction introduced by noncommutativity in the entropy of the BTZ black hole. Again in the limit $\theta \rightarrow 0$ we recover the entropy of the commutative BTZ black hole i.e.

$$S = 4\pi r_+.$$

If we plot the mass of the noncommutative BTZ black hole in Eq. (38) for different values of $\beta\theta$, we obtain the graph as given in Fig. 1.

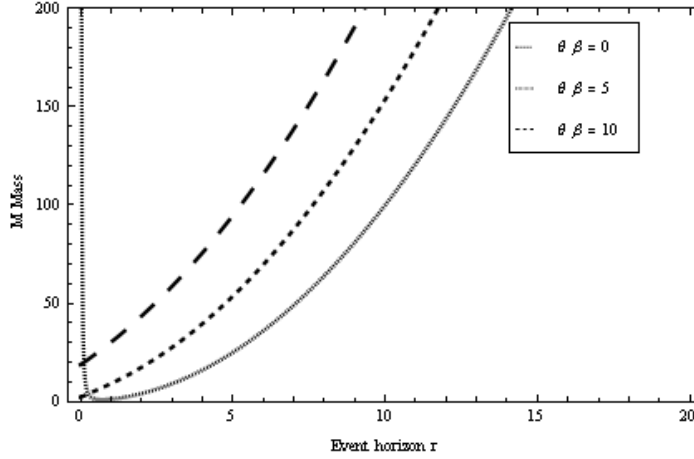


Figure 1: Mass of the noncommutative BTZ black hole versus the event horizon of the commutative BTZ black hole for different values of $\beta\theta$, while setting $J = l = Q = 1$.

The graph shows that there is an increase in the mass of noncommutative BTZ black hole when $\beta\theta$ is increasing from 0 to 10. For the commutative case the mass becomes infinite when we take limit $r \rightarrow 0$ but in the noncommutative case it remains finite. Now if we plot the temperature of the noncommutative BTZ black hole, given in Eq. (39) for different values of $\beta\theta$, we get the graph as in Fig. 2.

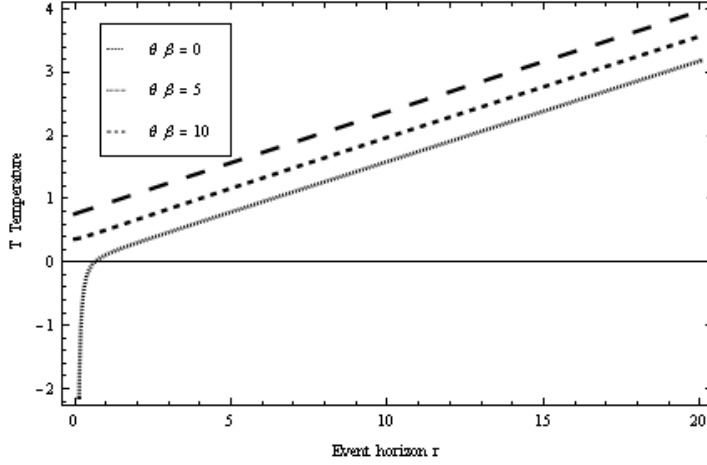


Figure 2: Temperature of the noncommutative BTZ black hole versus the event horizon of the commutative BTZ black hole for different values of $\beta\theta$, while setting $J = l = Q = 1$.

The entropy from Eq. (52) for different values of $\beta\theta$ is given in Fig. 3. The graph shows that there is an increase in the entropy of noncommutative BTZ black hole when $\beta\theta$ increases from 0 to 10. In the commutative case the entropy at the origin is zero but in the noncommutative case it is nonzero.

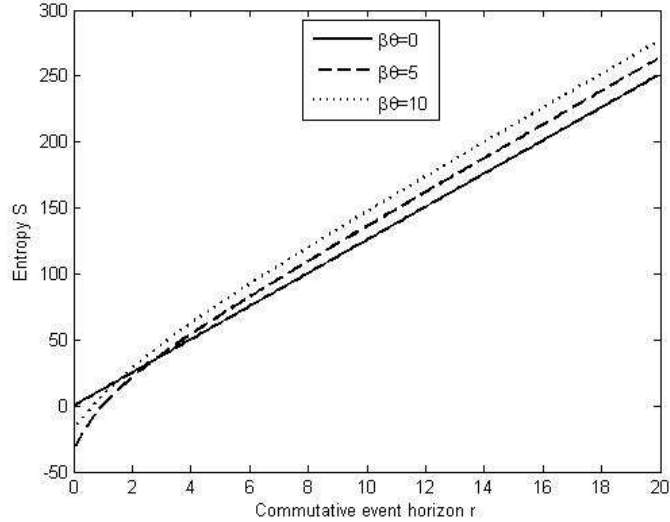


Figure 3: Entropy of the noncommutative BTZ black hole versus the event horizon of the commutative BTZ black hole for different values of $\beta\theta$ while setting $J = l = Q = 1$.

We know that in commutative geometry the entropy of a black hole is proportional to its area. The question is whether this area law holds for noncommutative black holes or not. In the case of noncommutative Schwarzschild black hole [1] the law is valid for the region $r_h \geq 3\sqrt{\theta}$, but for the region $r_h < 3\sqrt{\theta}$ this law is not valid.

The area formula for the BTZ black hole is [8]

$$A = 16\pi G_3 r_+, \quad (53)$$

where G_3 is the three dimensional Newton's gravitational constant. Using entropy from Eq. (52), we see that the area for the noncommutative case takes the form

$$A = 16\pi\hbar G_3 \left(r_+ + \frac{\beta\theta}{4} - \frac{\beta\theta}{r_+ + \beta\theta/4} \right). \quad (54)$$

The entropy (as in Eq. (51)) for the noncommutative BTZ black hole is plotted against the event horizon for different values of $\beta\theta$ in Figs. 4 and 5. We see that the area law holds for large r but as it approaches zero the law no longer remains valid.

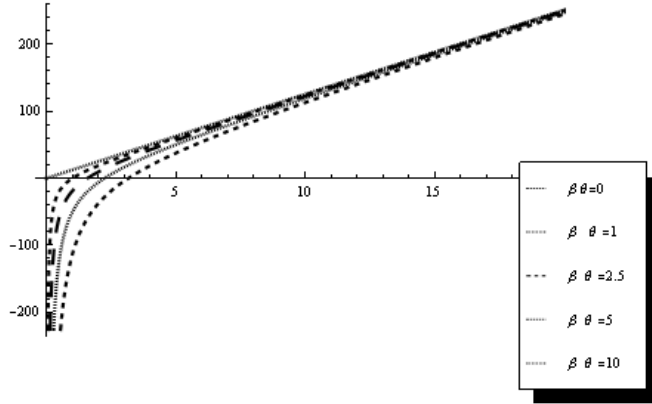


Figure 4: Entropy of the noncommutative BTZ black hole versus the event horizon of the noncommutative BTZ black hole

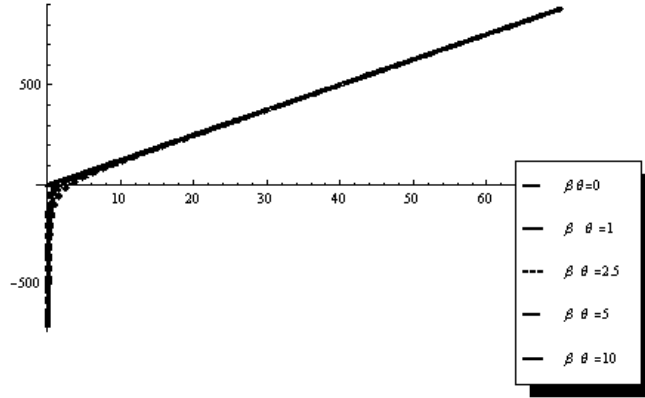


Figure 5: Entropy of the noncommutative BTZ black hole versus the event horizon of the noncommutative BTZ black hole (this is in fact a zoom-out of Fig. 4.)

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